Journal of Statistical Physics, Vol. 29, No. 3, 1982

Crystallographic Groups and Homogeneous Statistical Solutions of Navier–Stokes Equations

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Received November 10, 1981

In this paper, we investigate the connection between crystallographic groups and homogeneous statistical solutions of Navier–Stokes equations. Several results of Foias and Temam are extended. Fluid flows invariant under crystallographic groups are studied. This idea may be of interest to the understanding of bifurcation and turbulence.

KEY WORDS: Homogeneous statistical solution; Navier–Stokes equations; crystallographic group; Euclidean motion group.

1. INTRODUCTION

In the mathematical theory of turbulent flows, statistical solutions to the Navier–Stokes equations

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0$$

$$\nabla \cdot u = 0$$

are a family $\{\mu_t\}$ of measures on a suitable function space H such that if the initial data satisfy

$$\mu_0(A) = \text{Prob}\{u_0 \in A\}, \qquad \text{for each Borel set} \quad A \subset H$$

then

$$\mu_t(A) = \text{Prob}\{u(t) \in A\} \qquad (t > 0)$$

and satisfy some conditions.

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0022-4715/82/1100-0579$03.00/0

This project has been carried out by Foias,[2,3] Foias and Temam,[4] Hopf,[6] and Vishik and Foursikov[9,10] and others (see above-mentioned papers for further references). In particular, Foias and Temam[4] and Vishik and Fouriskov,[9,10] have investigated the homogeneous turbulence or homogeneous statistical solutions to the Navier–Stokes equations. The equations studied by Vishik and Foursikov are modified Navier–Stokes equations with some perturbed terms.

A measure μ on a suitable function space in $L^2_{\mathrm{loc}}(\mathbb{R}^3)^3$ is homogeneous if μ is invariant under all the induced actions of translations of $\mathbb{R}^3$. In Ref. 4, Foias and Temam have proved the important result that homogeneous statistical solutions or homogeneous measures satisfying certain conditions exist for the given initial homogeneous measure of the Navier–Stokes equations (Theorem 4.1, p. 27 of Ref. 4). The idea of the proof is that any homogeneous statistical solution is a limit in some weak sense of homogeneous statistical solutions concentrated on flows periodic in the space variables. The existence of homogeneous statistical solutions for periodic flows can be proved using function spaces on cubes which are compact.

In this paper, we shall extend the definition of homogeneous measure and of the definition of homogeneous statistical solution of Foias and Temam (or Vishik and Foursikov) to the full group $E(3)$ of motions of the Euclidean 3-space $\mathbb{R}^3$ and extend the periodicity to invariance with respect to a discrete subgroup of $E(3)$ with compact fundamental domain (or a crystallographic subgroup). We are able to extend the result of Foias and Temam on periodic flows to this setting which seems to be natural under physical and experimental consideration. The relation of crystallographic subgroups of $E(n)$ with stability and bifurcation of fluid dynamics is described in Ref. 7.

This paper is organized as follows. In Section 2, we present basic concepts about Euclidean motion group, discrete subgroups, fundamental domains, Bieberbach's celebrated structure theorem, and classification of crystallographic groups. This section is not so well known to experts in fluid mechanics. We define, in Section 3, homogeneous measure and introduce some familiar function spaces. In Section 4, periodic flows are extended to flows invariant under crystallographic groups.

We follow closely the arguments of Foias and Temam.[4] Our main idea is to use a bigger group $E(3)$ instead of the translations. The full group of motions is the semidirect product $SO(3) \cdot \mathbb{R}^3$, where $SO(3)$ denotes the rotation group and $\mathbb{R}^3$ denotes the translation group. According to Bieberbach,[11] every crystallographic group has a normal subgroup of finite index in the translation group $\mathbb{R}^3$ and any minimal set of generators of the normal subgroup is a vector space basis of $\mathbb{R}^3$ relative to which the $SO(3)$ components of the elements of the crystallographic group have all entries integral. There are only finitely many isomorphism classes of crystallographic groups on $\mathbb{R}^3$.

We mention several situations where our arguments can be extended. Let us consider a bounded domain D which has a smaller group of motions than $E(3)$. In many cases, the group of motions may be only the identity. The half-space $\mathbb{R}^2 \times \mathbb{R}^+$ has a quite large group which includes $E(2)$ as a proper subgroup. The cylinder has also a nontrivial group of motions. We can consider discrete subgroups of the group of motions and carry out the same arguments.

2. THE EUCLIDEAN MOTION GROUP AND CRYSTALLOGRAPHIC SUBGROUPS

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual metric. By a motion of $\mathbb{R}^n$, we mean a nonhomogeneous linear transformation g which preserves both the distance between points of $\mathbb{R}^n$ and its orientation. It is known that any motion g in $\mathbb{R}^n$ can be written in the form

$$x \to Ax + a$$

where A is a rotation in $\mathbb{R}^n$ [that is, some element of the group $SO(n)$], and a is a vector of $\mathbb{R}^n$. We shall write

$$g = (A, a)$$

The group operation is defined by

$$(A_1, a_1)(A_2, a_2) = (A_1A_2, A_1a_2 + a_1)$$

In particular,

$$(A, a) = (I, a)(A, 0) = (A, 0)(I, A^{-1}a)$$

where I denotes the identity rotation.

It is easy to verify that an invariant measure in the n-dimensional Euclidean motion group $E(n)$ is given by $dg = dA \cdot da$, where dA is the normalized invariant measure in $SO(n)$ and da is the Euclidean measure in $\mathbb{R}^n$.

Let G be a Hausdorff topological group. A discrete subgroup is a subgroup which is a discrete subset. If H is a closed subgroup of G, then the coset space G/H has the quotient topology for the projection $G \to G/H$.

A group Γ of homeomorphisms acting on a connected locally arcwise connected space X is discontinuous at a point x if given any sequence $\{\gamma_i\} \subset \Gamma$ of distinct elements, the sequence $\{\gamma_i(x)\} \in X$ has no accumulation point. Γ is discontinuous on X if it is discontinuous at every point x of

X. Γ is properly discontinuous on X if every point x in X has a neighborhood U such that $\{\gamma \subset \Gamma : \gamma(U) \text{ meets } U\}$ is finite.

Proposition 2.1. Let Γ and K be subgroups of G with K compact and G locally compact. The following are equivalent: (1) Γ is discontinuous at some point of G/K, (2) Γ is discontinuous on G/K, (3) Γ is properly discontinuous on G/K, (4) Γ is discrete in G.

Proof. See p. 99 of Ref. 11.

Now, we apply the above general facts to the case $G = E(n)$, $K = SO(n)$ and Γ is a discrete subgroup of $E(n)$.

$\mathbb{R}^n$ can be identified with $E(n)/SO(n)$. Thus, we have that a subgroup Γ of $E(n)$ acts properly discontinuously on $\mathbb{R}^n$ if and only if Γ is discrete in $E(n)$. A closed subgroup H of G is uniform if G/H is compact. Then Γ is a uniform discrete subgroup of $E(n)$ if and only if Γ acts properly discontinuously with compact quotient on $\mathbb{R}^n$. A discrete uniform subgroup of $E(n)$ is called a crystallographic group on $\mathbb{R}^n$. This name came from the study of crystalline structures on $\mathbb{R}^n$.

The following theorem of Bieberbach is fundamental.

Theorem 2.2 (Bieberbach). If $\Gamma \cap E(n)$ is a crystallographic group, then $\Gamma \cap R^n$ is a normal subgroup of finite index in Γ, and any minimal set of generators of $\Gamma \cap \mathbb{R}^n$ is a vector space basis of $\mathbb{R}^n$ relative to which the $SO(n)$ components of the elements of Γ have all entries integral. For each integer $n > 0$, there are only a finite number of isomorphism classes of crystallographic groups on $\mathbb{R}^n$.

The Euclidean space form problem is the classification of all discrete groups of motions which are fixed point free. This problem has only been completed recently by Wolf[11] and others (see Ref. 11 for references). For $\mathbb{R}^2$, the problem is quite easy, namely, the Euclidean 2-space forms are the Euclidean plane, the cylinders, and the tori. If we allow nonorientation preserving motions such as in Ref. 11, we have, in addition, the Moebius bands and the Klein bottles. The case of three-dimensional Euclidean space forms are much more difficult. The noncompact case is given by Wolf (Theorem 3.5.1 of Ref. 11). We shall state the case of compact orientable Euclidean 3-space forms, because it is related to the study of the Navier–Stokes equations.

Theorem 2.3 (Hantzsche and Wendt[11]). There are just six affine diffeomorphism classes of compact connected orientable Euclidean 3-space forms. The group Γ is one of the following six groups. Here $\{a_1, a_2, a_3\}$ is a translation lattice and $t_i = t_{a_i}$ is the corresponding translation.

(1) Γ is generated by the translations $\{t_1, t_2, t_3\}$ with $\{a_i\}$ linearly independent.

(2) Γ is generated by $\{\alpha, t_1, t_2, t_3\}$, where $\alpha^2 = t_1$, $\alpha t_2 \alpha^{-1} = t_2^{-1}$, and $\alpha t_3 \alpha^{-1} = t_3^{-1}$; a_1 is orthogonal to a_2 and a_3, while $\alpha = (A, a_1/2)$ with $A(a_1) = a_1, A(a_2) = -a_2, A(a_3) = -a_3$.

(3) Γ is generated by $\{\alpha, t_1, t_2, t_3\}$, where $\alpha^3 = t_1$, $\alpha t_2 \alpha^{-1} = t_3$ and $\alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1}$; a_1 is orthogonal to a_2 and a_3, $\|a_2\| = \|a_3\|$, and $\{a_2, a_3\}$ is a hexagonal plane lattice, and $\alpha = (A, a_1/3)$ with $A(a_1) = a_1$, $A(a_2) = a_3$ and $A(a_3) = -a_2 - a_3$.

(4) Γ is generated by $\{\alpha, t_1, t_2, t_3\}$, where $\alpha^4 = t_1$, $\alpha t_2 \alpha^{-1} = t_3$, and $\alpha t_3 \alpha^{-1} = t_2^{-1}$; $\{a_i\}$ are mutually orthogonal with $\|a_2\| = \|a_3\|$, while $\alpha = (A, a_1/4)$ with $A(a_1) = a_1$, $A(a_2) = a_3$ and $A(a_3) = -a_2$.

(5) Γ is generated by $\{\alpha, t_1, t_2, t_3\}$, where $\alpha^6 = t_1$, $\alpha t_2 \alpha^{-1} = t_3$, and $\alpha t_3 \alpha^{-1} = t_2^{-1} t_3$; a_1 is orthogonal to a_2 and a_3, $\|a_2\| = \|a_3\|$, and $\{a_2, a_3\}$ is a hexagonal plane lattice, and $\alpha = (A, a_1/6)$ with $A(a_1) = a_1, A(a_2) = a_3$, and $A(a_3) = a_3 - a_2$.

(6) Γ is generated by $\{\alpha, \beta, \gamma; t_1, t_2, t_3\}$, where $\gamma\beta\alpha = t_1 t_3$ and

$$\alpha^2 = t_1, \qquad \alpha t_2 \alpha^{-1} = t_2^{-1}, \qquad \alpha t_3 \alpha^{-1} = t_3^{-1}$$

$$\beta t_1 \beta^{-1} = t_1^{-1}, \qquad \beta^2 = t_2, \qquad \beta t_3 \beta^{-1} = t_3^{-1}$$

$$\gamma t_1 \gamma^{-1} = t_1^{-1}, \qquad \gamma t_2 \gamma^{-1} = t_2^{-1} \gamma^2 = t_3$$

The $\{a_i\}$ are mutually orthogonal and $\alpha = (A, a_1/2)$, $\beta = [B, (a_2 + a_3)/2]$, and $\gamma = [C, (a_1 + a_2 + a_3)/2]$, with $A(a_1) = a_1$, $A(a_2) = -a_2$, $A(a_3) = -a_3$; $B(a_1) = -a_1$, $B(a_2) = a_2$, $B(a_3) = -a_3$; $C(a_1) = -a_1$, $C(a_2) = -a_2$, $C(a_3) = a_3$.

Let X be a topological space on which a discrete group Γ of homeomorphisms acts. We introduce the concept of a fundamental domain in X relative to Γ. A fundamental domain in X relative to Γ is defined as an open set $F \subset X$ satisfying the following two conditions:

1. For arbitrary $\gamma_1 \neq \gamma_2$ the sets $\gamma_1 F$ and $\gamma_2 \bar{F}$, where $\bar{F}$ is the closure of F, have no common elements.
2. The union of the sets $\gamma \bar{F}$, where γ ranges over Γ, is the whole space X.

Basically, every point x of X can be represented in the form γx, where γ in Γ and $x \in \bar{F}$. This representation is unique for almost all points in X. The nonuniqueness may only occur at $\Gamma(\bar{F} \backslash F)$. A fundamental domain relative to a discrete group Γ is not uniquely determined by the group Γ, in fact, if F is a fundamental domain then every translation γF of it, $\gamma \in \Gamma$, is also a fundamental domain.

There is a well-known method of constructing a fundamental domain for a locally compact metric space X satisfying the following: for any two points x_1 and x_2 one can find a third pont x_3 such that $d(x_1, x_3) = d(x_3, x_2) = (1/2) d(x_1, x_2)$. For Euclidean space $\mathbb{R}^n$, this condition is satisfied. We assume that the group Γ preserves the metric d or they are isometries. We select a point x_0 that is not a fixed point. We consider the set F of those points x such that $d(x_0, x) < d(\gamma x_0, x)$ for every $\gamma \neq e$ in Γ.

It is not difficult to prove that the above defined F is a fundamental domain relative to Γ. There is much interest in discrete groups for which the closure $\bar{F}$ of a fundamental domain F is a compact set. In our case of interest, $\bar{F}$ being compact is equivalent to $\bar{F}$ being a closed and bounded set. One can prove that under the compactness assumption there are finitely many $\sigma_1, \ldots, \sigma_m$ of Γ such that $\bar{F}$ can be given by finitely many inequalities

$$d(x_0, x) \leqslant d(\sigma_i x_0, x), \qquad i = 1, \ldots, m$$

In other words, Γ is finitely generated and has finitely many relations among the generators.

An example of the above discussion is that Γ is a discrete subgroup of a locally compact group G such that $\Gamma \backslash G$ is compact. Then there exists a fundamental domain F in G relative to Γ and the closure $\bar{F}$ of F is compact. In this special case, we shall write C instead of F and write $G = \Gamma \cdot C$. We shall apply this notation for the Euclidean motion group and its crystallographic subgroups.

3. FUNCTION SPACES AND HOMOGENEOUS MEASURES

Let $L^2(\mathbb{R}^n)$ denote the space of square integrable real functions on $\mathbb{R}^n$ and $H^m(\mathbb{R}^n)$ denote the Sobolev space of functions in $L^2(\mathbb{R}^n)$ together with their derivatives of order $\leqslant m$. The space $L^2_{\text{loc}}(\mathbb{R}^n)$ [respectively, $H^m_{\text{loc}}(\mathbb{R}^n)$] is the space of functions which are locally L^2 (respectively, H^m) on $\mathbb{R}^n$. $\mathbf{L}^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)^n$, $\mathbf{L}^2_{\text{loc}}(\mathbb{R}^n) = L^2_{\text{loc}}(\mathbb{R}^n)^n$, $\mathbf{H}^m(\mathbb{R}^n) = H^m(\mathbb{R}^n)^n$, $\mathbf{H}^m_{\text{loc}}(\mathbb{R}^n) = H^m_{\text{loc}}(\mathbb{R}^n)^n$.

If Q is a bounded measurable subset of $\mathbb{R}^n$ and u is in $\mathbf{L}^2_{\text{loc}}(\mathbb{R}^n)$, then

$$|u|_Q = \left[\frac{1}{|Q|} \int_Q |u(x)|^2 \, dx \right]^{1/2}$$

$$\|u\|_Q = \left\{ \frac{1}{|Q|} \int_Q \left[\sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j}(x) \right|^2 \right] dx \right\}^{1/2}$$

and

$$|u|_{m,Q} = \left\{ \frac{1}{|Q|} \int_Q \left[\sum_{|\alpha| \leqslant m} |D^\alpha u(x)|^2 \right] dx \right\}^{1/2},$$

where $|Q|$ = measure of Q.

Let

$$H_{\text{loc}} = \left\{ u \in \mathbf{L}^2_{\text{loc}}(\mathbb{R}^n), \nabla \cdot u = 0 \right\}$$

$$V_{\text{loc}} = \left\{ u \in \mathbf{H}^1_{\text{loc}}(\mathbb{R}^n), \nabla \cdot u = 0 \right\}$$

be the Frechet spaces endowed with the above seminorms.

For each g in $E(n)$, we denote by τ_g the operator defined on all function spaces on $\mathbb{R}^n$ by

$$(\tau_g u)(x) = u(gx)$$

If Γ is a crystallographic subgroup of $E(n)$ and $F(\Gamma)$ is a fundamental domain relative to Γ in $\mathbb{R}^n$, then

$$H(\Gamma) = \left\{ u \in H_{\text{loc}}, \tau_\gamma u = u, \text{ all } \gamma \in \Gamma \right\}$$

$$V(\Gamma) = \left\{ u \in V_{\text{loc}}, \tau_\gamma u = u, \text{ all } \gamma \in \Gamma \right\}$$

These spaces are Hilbert spaces for the norms $|u|_{F(\Gamma)}$ and $|u|_{1,F(\Gamma)}$.

We consider probability measures on H_{loc}, that is, Borel measures on H_{loc} which are positive and of mass 1. Let g be in $E(n)$ and let τ_g be the operator which is linear and continuous on H_{loc} to itself. If μ is a probability measure on H_{loc}, then the image measure $\tau_g(\mu)$ is well defined and is also a probability measure on H_{loc}. This is just the induced action on the space of all probability measures on H_{loc}. We still use the same notation τ_g for the induced action.

Definition 3.1. A probability measure μ on H_{loc} is homogeneous if $\tau_g(\mu) = \mu$ for every g in $E(n)$.

Thus, every translation-invariant measure[4,9] is not homogeneous in our sense and our homogeneity includes rotational invariance also.

A measure μ being homogeneous is equivalent to that for every function $\phi \in \mathrm{B}(H_{\text{loc}})$, the space of real bounded continuous functions on H_{loc}, we have

$$\int \phi(\tau_g u)\, d\mu(u) = \int \phi(u)\, d\mu(u)$$

for every g in $E(n)$.

The two lemmas of Ref. 4 on homogeneous measure are valid.

Lemma 3.1. Let μ be a homogeneous measure on H_{loc} and let G be a mapping defined $d\mu$ a.e. from H_{loc} into $L^1_{\text{loc}}(\mathbb{R}^n)$ such that the mapping $u \to \int_Q G(u)(x)\, dx$ is $d\mu$ measurable for all compact subsets. We assume that G commutes with the $E(n)$ actions: $G \circ \tau_g = \tau_g \circ G$, for all g in $E(n)$, and for some compact subset Q_0,

$$\int\int_{Q_0} |G(u)(x)|\, dx\, d\mu(u) < +\infty$$

Then for every $\phi \in L^1(R)$, the integral

$$\int\int G(u)(x)\phi(x)\, dx\, d\mu(u)$$

makes sense and equals $\langle G\rangle \int \phi\, dx$, where $\langle G\rangle$ is a constant independent of ϕ, which is denoted by

$$\int G(u)(x)\, d\mu(x) \qquad \text{for every } x \text{ in } R^n$$

If $G(u)(x) = A(D)F(u)(x)$, where $A(D)$ is a differential operator with no zero-order term and F maps H_{loc} into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies the same conditions as G, then

$$\int G(u)(x)\, d\mu(u) = 0$$

Lemma 3.2. Under the same assumptions as the previous lemma, and if Q is a bounded measurable set in $\mathbb{R}^n$, the value of the integral

$$\int \frac{1}{|Q|} \int_Q G(u)(x)\, dx\, d\mu(u)$$

is independent of Q.

Let H be a Hilbert space. The real function ϕ defined on H by

$$\phi(u) = e^{-1/2(Su,u)_H}$$

where S is a linear continuous operator on H, is the characteristic function of a Gaussian probability measure if and only if

$$S \geqslant 0 \quad \text{and} \quad \operatorname{Tr} S < +\infty$$

We define a Gaussian probability measure on $H = H(\Gamma)$ this way.

Let Γ be a crystallographic group with $E(n) = \Gamma \cdot C$, where C is a compact fundamental domain relative to Γ in $E(n)$. Then we have the following.

Lemma 3.3. If μ is a probability measure on H_{loc}, concentrated on $H(\Gamma)$, then

$$\tilde{\mu} = \frac{1}{|C|} \int_C \tau_g(\mu)\, dg$$

where $|C|$ denotes the Haar measure of C in $E(n)$, is a homogeneous probability measure concentrated on $H(\Gamma)$.

We can follow the argument of Ref. 4 to prove this lemma or apply the fixed-point property of amenable groups acting on compact convex subsets in locally convex topological vector spaces. The Euclidean group is known to be amenable. The space of probability measures on H_{loc} is compact and convex in a locally convex topological vector space. The fixed point will be the homogeneous measure that we are looking for.

Proof. We have to show that $\tilde{\mu}$ is homogeneous. If $\phi \in \mathrm{B}(H_{\text{loc}})$ and $g \in E(n)$, we need

$$\int \phi(\tau_g \mu)\, d\tilde{\mu}(u) = \int \frac{1}{|C|} \int_C \phi(\tau_{gg'}, u)\, dg'\, d\mu(u)$$

But $\phi(\tau_g u)$, for $u \in H(\Gamma)$, is Γ invariant. Thus for $d\mu$ a.e.

$$\int_C \phi(\tau_{gg'}, u)\, dg' = \int_{gC} \phi(\tau_{g'}, u)\, dg' = \int_C \phi(\tau_g u)\, dg$$

by the Γ invariance of ϕ and $gC = \cup_{i=1}^{N} (\gamma_i C \cap gC)$ for some γ_i, $1 \leqslant i \leqslant N$ and

$$\int \phi(\tau_g u)\, d\tilde{\mu}(u) = \int \phi(u)\, d\tilde{\mu}(u)$$

for every g in $E(n)$. ■

4. STATISTICAL SOLUTIONS OF NAVIER–STOKES EQUATIONS

The problem is to find a vector function $u = (u_1, \ldots, u_n)$ and a scalar function p defined on $\mathbb{R}^n \times (0, T)$ ($T > 0$ is arbitrarily large) such that

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \qquad & \text{in} \quad \mathbb{R}^n \times (0, T) \\ \nabla u = 0 \qquad & \text{in} \quad \mathbb{R}^n \times (0, T) \\ \tau_\gamma u = u, \qquad & \gamma \in \Gamma \\ u(x, 0) = u_0(x), \qquad & u_0 \text{ given in } H(\Gamma) \end{aligned} \tag{4.1}$$

where $\nu > 0$, Γ is given, f is the given body force and $f \in L^2(0, T; H(\Gamma))$.

If $n = 2$, for u_0 and f given in $H(\Gamma)$ and $L^2(0, T; H(\Gamma))$, there exists a unique solution u in $L^2(0, T; V(\Gamma)) \cap C([0, T]; H(\Gamma))$. If $n = 3$, we know the existence of a solution u in $L^2(0, T; V(\Gamma)) \cap C([0, T]; H(\Gamma)_W)$, where $H(\Gamma)_W$ denotes $H(\Gamma)$ with the weak topology.

Definition 4.1. A statistical solution of (4.1) is a family of measures $\{\mu_t, 0 < t < T\}$ on $H(\Gamma)$ which satisfies the following conditions:

(1) For every $\phi \in \mathsf{B}(H(\Gamma))$, the mapping $t \to \int \phi(u)\, d\mu_t(u)$ is measurable:

(2) $\int |u|^2_{F(\Gamma)}\, d\mu_t(u) \in L^\infty(0,T)$, $\int \|u\|^2_{F(\Gamma)}\, d\mu_t(u) \in L^1(0,T)$

(3)
$$\int \phi(u)\, d\mu_t(u) + |F(\Gamma)| \times \int_0^t \left\{ \nu\left[(u, \phi'(u)) \right]_{F(\Gamma)} + \langle B(u), \phi'(u) \rangle_{F(\Gamma)} \right\} d\mu_s(u)\, ds = |F(\Gamma)| \int_0^t \int \left[f(s), \phi'(u) \right]_{F(\Gamma)} d\mu_s(u)\, ds + \int \phi(u)\, d\mu_0(u)$$

for $t \in [0, T]$, $\phi \in P_0$.

(4)
$$\int |u|^2_{F(\Gamma)}\, d\mu_t(u) + 2\nu \int_0^t \int \|u\|^2_{F(\Gamma)} d\mu_s(u)\, ds \leqslant \int_0^t \left[f(s), u(s) \right]_{F(\Gamma)} d\mu_s(u)\, ds + \int |u|^2_{F(\Gamma)}\, d\mu_0(u)$$

for $t \in [0, T]$.

Here P_0 is the family of functions $\phi(u) = \psi[(u, g_1)_{F(\Gamma)}, \ldots, (u, g_k)_{F(\Gamma)}]$, where ψ is C^1 on $\mathbb{R}^k$ with compact support and $g_1, \ldots, g_k \in V(\Gamma)$. $\langle \cdot, \cdot \rangle_{F(\Gamma)}$ is the pairing between $V(\Gamma)$ and $V'(\Gamma)$ and $B(u) = B(u,u)$, where $B(u,v)$ is the operator from $V(\Gamma) \times V(\Gamma)$ into $V'(\Gamma)$ defined by

$$\langle B(u,v), w \rangle = \frac{1}{|F(\Gamma)|} \int_{F(\Gamma)} \left[u \cdot \nabla)v \right] w\, dx$$

for $u, v, w \in V(\Gamma)$. ϕ' is the differential of ϕ in $H(\Gamma)$.

The main theorem in this section is the following:

Theorem 4.1. Given f and μ_0, $f \in L^2(0,T; H(\Gamma))$, and μ_0 a probability measure on $H(\Gamma)$, $T > 0$ arbitrarily large, there exists a family of measures $\{\mu_t\}$ $0 \leqslant t \leqslant T$ which satisfies the conditions of the definition of statistical solutions in Definition 4.1. Furthermore if $f = 0$ and μ_0 is a homogeneous probability measure concentrated on $H(\Gamma)$ then there exists a family $\{\mu_t\}$ $0 < t < T$ such that μ_t is concentrated on $H(\Gamma)$ and is homogeneous for every t.

The proof of this theorem needs some changes but essentially is the same as Ref. 4. The first step is the Galerkin method. The procedure is the same by replacing the cube $Q(L)$ of Ref. 4 by the compact fundamental domain $F(\Gamma)$. By the local solvability, we get a sequence $\mu_t^{(m)}$ of probability measures. The second step is the passage to the limit as m approaches infinity. This is the same as Ref. 4. The third step is the homogeneity. We are given the homogeneity of μ_0, however, we have to prove the homogeneity of μ_t. For every $t > 0$ we define

$$\tilde{\mu}_t = \frac{1}{|C|} \int_C \tau_g(\mu_t)\, dg, \qquad 0 \leqslant t \leqslant T$$

By Lemma 3.2, $\tilde{\mu}_t$ is a homogeneous probability measure concentrated on $H(\Gamma)$ and $\tilde{\mu}_0 = \mu_0$. The conditions in Definition 4.1 are satisfied, by using the relations

$$|u|^2_{F(\Gamma)} = |\tau_g u|^2_{F(\Gamma)}$$

$$\|u\|^2_{F(\Gamma)} = \|\tau_g u\|^2_{F(\Gamma)}$$

[for $u \in V(\Gamma)$ and $g \in E(n)$]

$$\phi_g = \phi \circ \tau_g$$

$$\phi'_g(u) = \phi'(\tau_g u) \circ \tau_g$$

$$[u, \phi'_g(u)]_{F(\Gamma)} = [\tau_g u, \phi'(\tau_g u)]_{F(\Gamma)}$$

$$\langle B(u), \phi'_g(u)\rangle_{F(\Gamma)} = \langle B(\tau_g u), \phi'(\tau_g u)\rangle_{F(\Gamma)}$$

ACKNOWLEDGMENTS

We would like to express our gratitude to Professors Foias and Temam for their paper[4] on which this study is based.

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